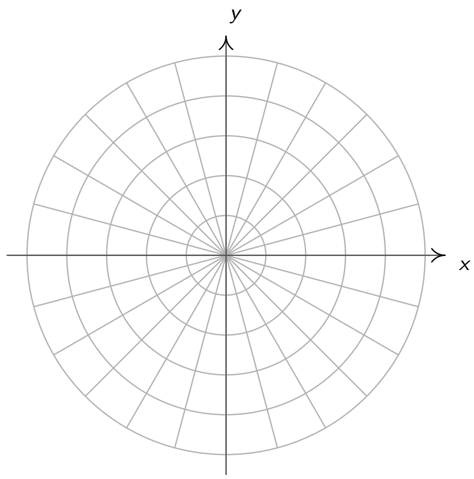
## MATH 1700: SECTION 13.2: GRAPHING IN POLAR COORDINATES

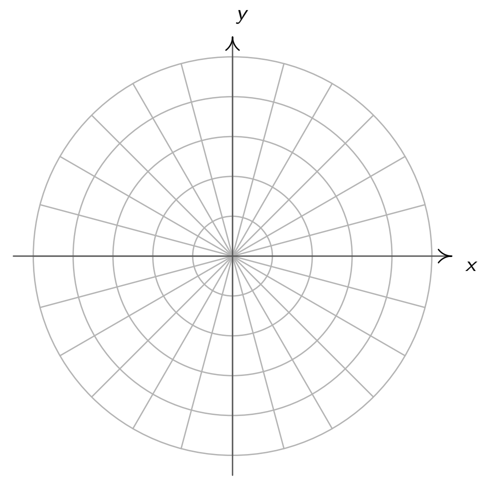
In this section, we discuss how to graph equations relating the *polar coordinate* variables $r$ and $\theta$ on the *rectangular coordinate* ($xy$–)plane. Since every point in the plane has infinitely many different representations in polar coordinates, in order for a point $P$ to be on the graph of a given equation, there must be *at least one* representation of $P(r, \theta)$ that satisfies that equation. In our first example, only one of the variables $r$ and $\theta$ is present making the other variable free. This makes these graphs easier to visualize than others.

**EXAMPLE 1:** Graph the following polar equations in the $xy$-plane.

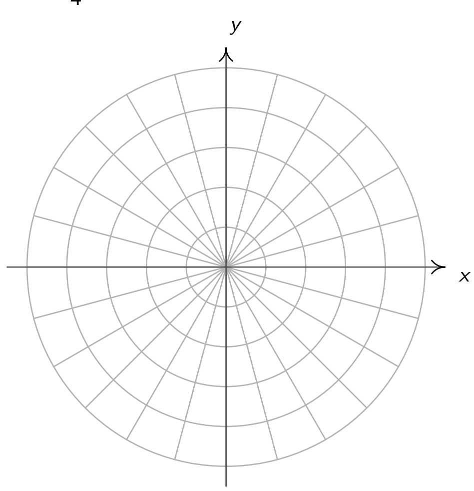
- $r = 4$



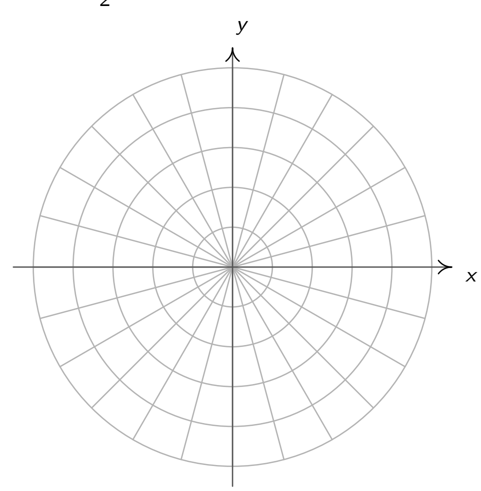

- $r = -3\sqrt{2}$




- $\theta = \frac{5\pi}{4}$




- $\theta = -\frac{3\pi}{2}$

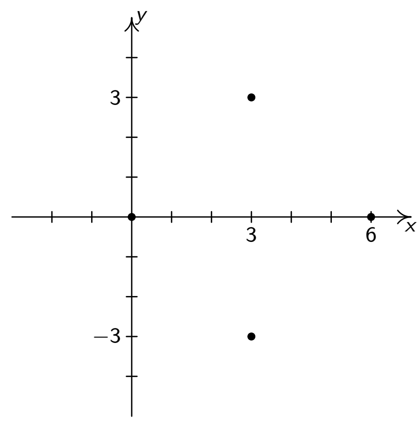


**GRAPHS OF CONSTANT $r$ AND $\theta$:** Suppose $a$ and $\alpha$ are constants, $a \neq 0$.

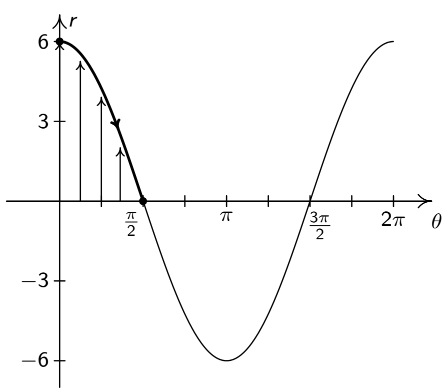
- The graph of the polar equation $r = a$ on the Cartesian plane is a circle centered at the origin of radius $|a|$.
- The graph of the polar equation $\theta = \alpha$ on the Cartesian plane is the line containing the terminal side of $\alpha$ when plotted in standard position.

Suppose we wish to graph $r = 6\cos(\theta)$. A reasonable way to start is to treat $\theta$ as the independent variable, $r$ as the dependent variable, evaluate $r = f(\theta)$ at some 'friendly' values of $\theta$ and plot the resulting points.

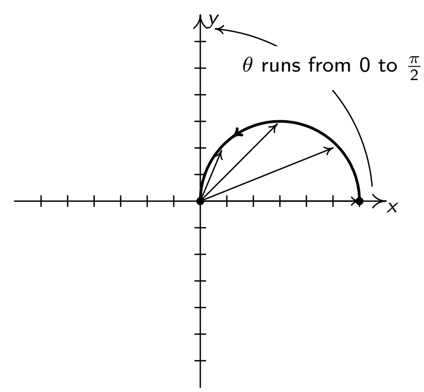
| $\theta$ | $r = 6\cos(\theta)$ | $(r, \theta)$ |
|---|---|---|
| $0$ | $6$ | $(6, 0)$ |
| $\frac{\pi}{4}$ | $3\sqrt{2}$ | $\left(3\sqrt{2}, \frac{\pi}{4}\right)$ |
| $\frac{\pi}{2}$ | $0$ | $\left(0, \frac{\pi}{2}\right)$ |
| $\frac{3\pi}{4}$ | $-3\sqrt{2}$ | $\left(-3\sqrt{2}, \frac{3\pi}{4}\right)$ |
| $\pi$ | $-6$ | $(-6, \pi)$ |
| $\frac{5\pi}{4}$ | $-3\sqrt{2}$ | $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ |
| $\frac{3\pi}{2}$ | $0$ | $\left(0, \frac{3\pi}{2}\right)$ |
| $\frac{7\pi}{4}$ | $3\sqrt{2}$ | $\left(3\sqrt{2}, \frac{7\pi}{4}\right)$ |
| $2\pi$ | $6$ | $(6, 2\pi)$ |



Despite having nine ordered pairs, we get only four distinct points on the graph. For this reason, we employ a slightly different strategy. We graph one cycle of $r = 6\cos(\theta)$ on the $\theta r$-plane below on the left and use it to help graph the equation on the $xy$-plane below on the right.



$r = 6\cos(\theta)$ in the $\theta r$-plane

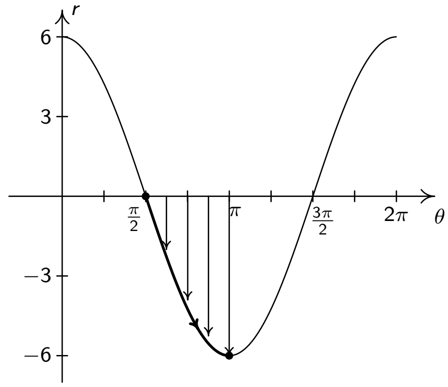



$r = 6\cos(\theta)$ in the $xy$-plane

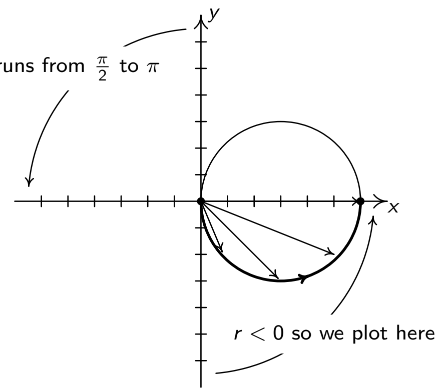
We see that as $\theta$ ranges from $0$ to $\frac{\pi}{2}$, $r$ ranges from $6$ to $0$. In the $xy$-plane, this means that the curve starts 6 units from the origin on the positive $x$-axis ($\theta = 0$) and gradually returns to the origin by the time the curve reaches the $y$-axis ($\theta = \frac{\pi}{2}$).

The arrows drawn in the figure below are meant to help you visualize this process. In the $\theta r$-plane, the arrows are drawn from the $\theta$-axis to the curve $r = 6\cos(\theta)$. In the $xy$-plane, each of these arrows starts at the origin and is rotated through the corresponding angle $\theta$, in accordance with how we plot polar coordinates. This method is less precise than plotting actual function values, but much faster.

Next, we repeat the process as $\theta$ ranges from $\frac{\pi}{2}$ to $\pi$. Here, the $r$ values are all negative. This means that in the $xy$-plane, instead of graphing in Quadrant II, we graph in Quadrant IV, with all of the angle rotations starting from the negative $x$-axis.



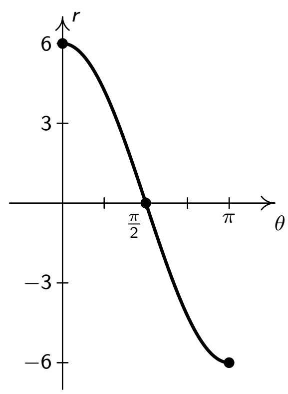
$r = 6\cos(\theta)$ in the $\theta r$-plane



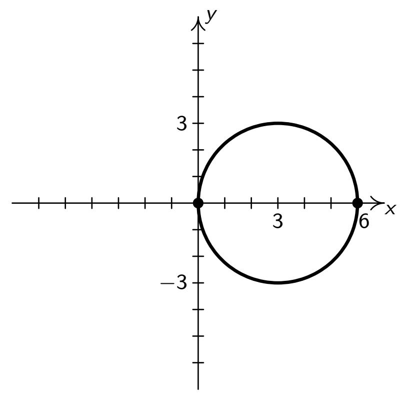
$r = 6\cos(\theta)$ in the $xy$-plane

As $\theta$ ranges from $\pi$ to $\frac{3\pi}{2}$, the $r$ values are still negative, which means the graph is traced out in Quadrant I instead of Quadrant III. Since the $|r|$ for these values of $\theta$ match the $r$ values for $\theta$ in $\left[0, \frac{\pi}{2}\right]$, we have that the curve begins to retrace itself at this point.

Proceeding further, we find that when $\frac{3\pi}{2} \leq \theta \leq 2\pi$, we retrace the portion of the curve in Quadrant IV that we first traced out as $\frac{\pi}{2} \leq \theta \leq \pi$. The reader is invited to verify that plotting any range of $\theta$ outside the interval $[0, \pi]$ results in retracting some portion of the curve. We present the final graph below.



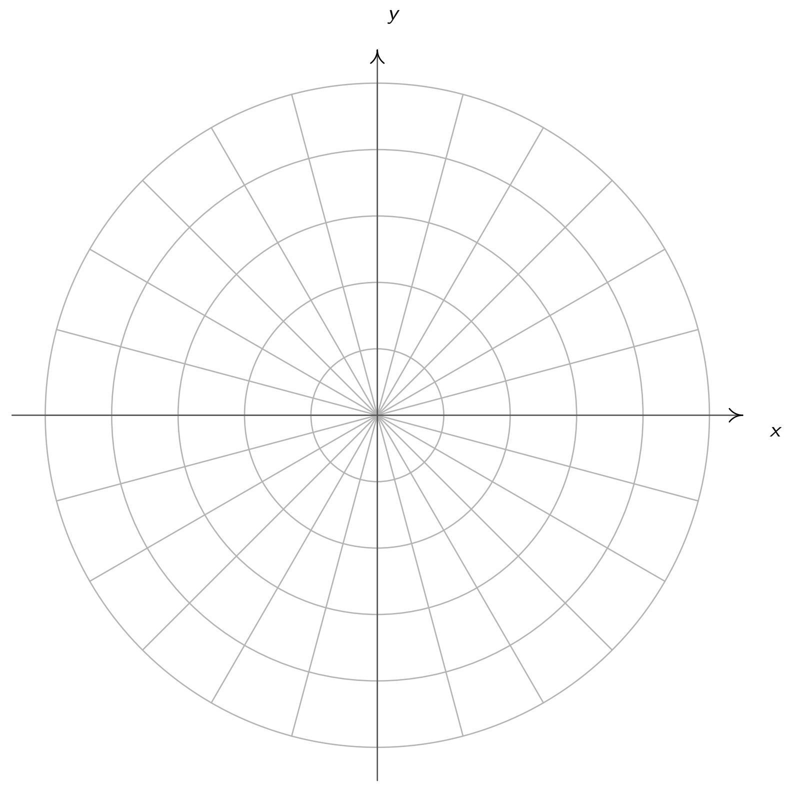
$r = 6\cos(\theta)$ in the $\theta r$-plane



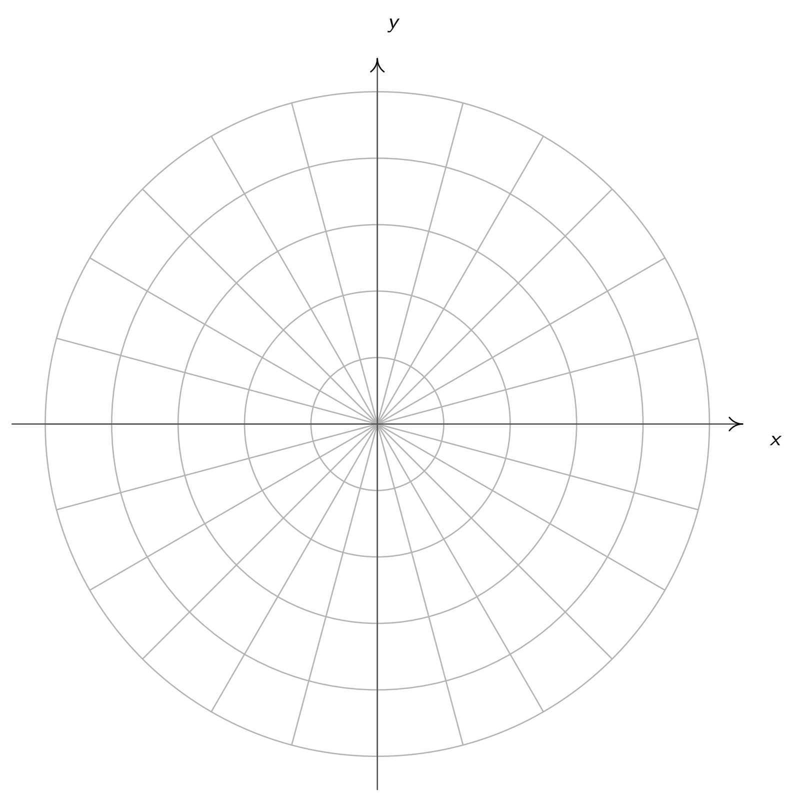
$r = 6\cos(\theta)$ in the $xy$-plane

**EXAMPLE 2:** Graph the following polar equations in the $xy$-plane.

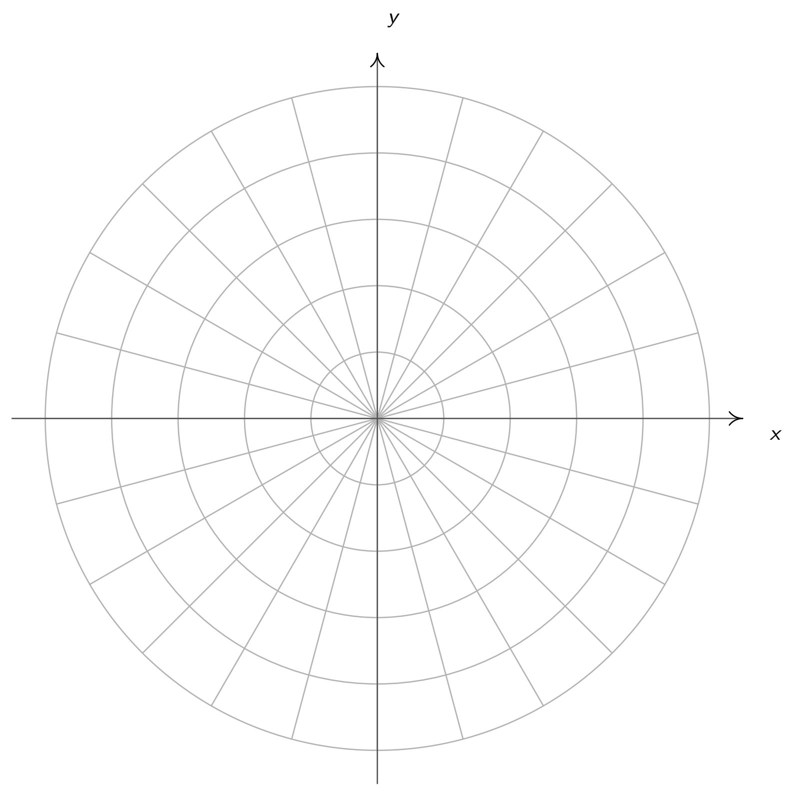
1. $r = 4 - 2\sin(\theta)$



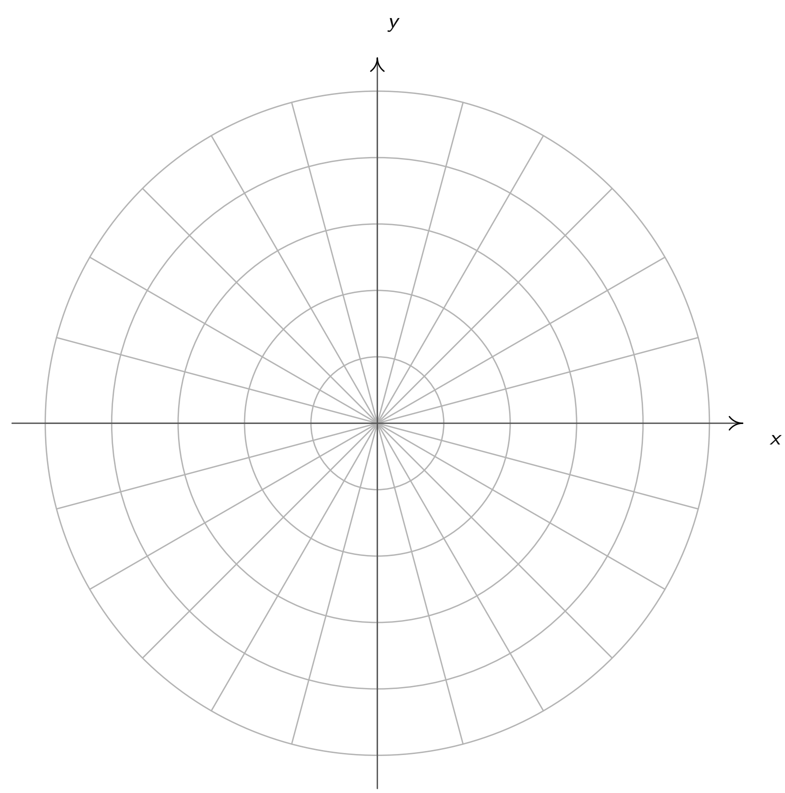

2. $r = 2 + 4\cos(\theta)$



3. $r = 5\sin(2\theta)$




4. $r^2 = 16\cos(2\theta)$



**EXAMPLE 3:** Find the points of intersection of the graphs of the following polar equations.

1. $r = 2\sin(\theta)$ and $r = 2 - 2\sin(\theta)$

2. $r = 2$ and $r = 3\cos(\theta)$

3. $r = 3$ and $r = 6\cos(2\theta)$

4. $r = 3\sin\left(\frac{\theta}{2}\right)$ and $r = 3\cos\left(\frac{\theta}{2}\right)$

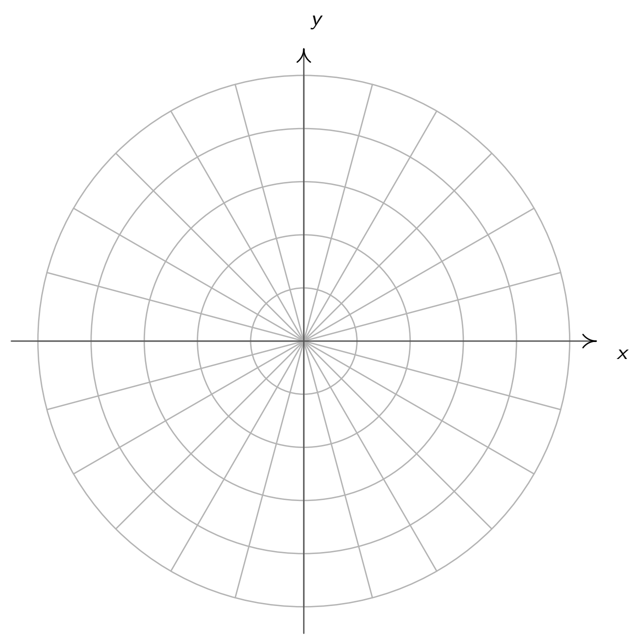
**GUIDELINES FOR FINDING POINTS OF INTERSECTION POINTS OF POLAR CURVES:**

To find the points of intersection of the graphs of two polar equations $E_1$ and $E_2$:

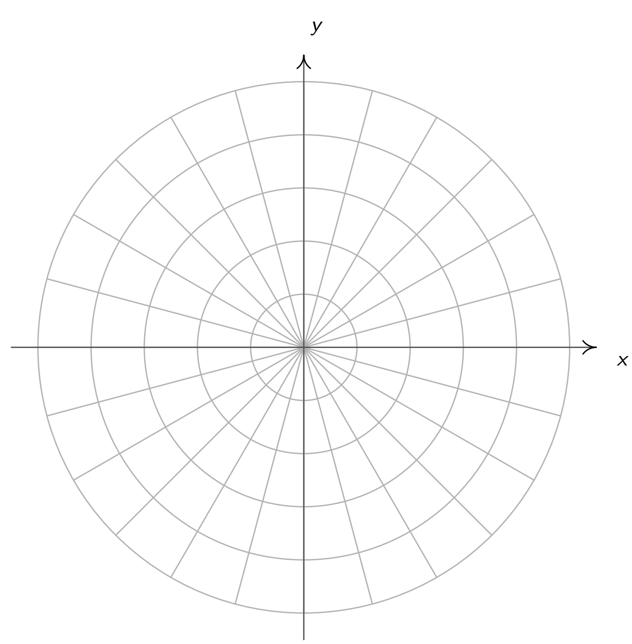
- Sketch the graphs of $E_1$ and $E_2$. Check to see if the curves intersect at the origin (pole).
- Solve for pairs $(r, \theta)$ which satisfy both $E_1$ and $E_2$.
- Substitute $(\theta + 2\pi k)$ for $\theta$ in either one of $E_1$ or $E_2$ (but not both) and solve for pairs $(r, \theta)$ which satisfy both equations. Keep in mind that $k$ is an integer.
- Substitute $(-r)$ for $r$ and $(\theta + (2k+1)\pi)$ for $\theta$ in either one of $E_1$ or $E_2$ (but not both) and solve for pairs $(r, \theta)$ which satisfy both equations. Keep in mind that $k$ is an integer.

**EXAMPLE 4:** Sketch the region in the $xy$-plane described by the following sets.

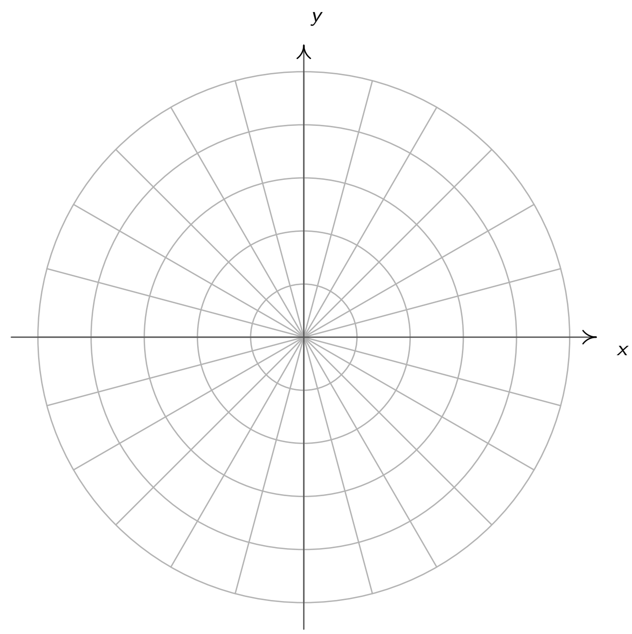
1. $\left\{(r,\theta)\,|\,0 \le r \le 5\sin(2\theta),\ 0 \le \theta \le \frac{\pi}{2}\right\}$



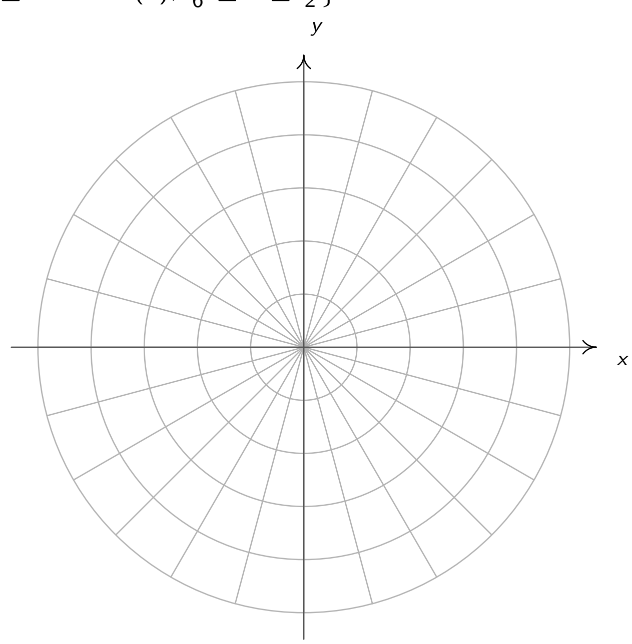

2. $\left\{(r,\theta)\,|\,3 \le r \le 6\cos(2\theta),\ 0 \le \theta \le \frac{\pi}{6}\right\}$



3. $\left\{(r,\theta)\,|\,2+4\cos(\theta)\leq r\leq 0,\ \frac{2\pi}{3}\leq\theta\leq\frac{4\pi}{3}\right\}$




4. $\left\{(r,\theta)\,|\,0\leq r\leq 2\sin(\theta),\ 0\leq\theta\leq\frac{\pi}{6}\right\}\cup\left\{(r,\theta)\,|\,0\leq r\leq 2-2\sin(\theta),\ \frac{\pi}{6}\leq\theta\leq\frac{\pi}{2}\right\}$




**HOMEWORK:** Section 13.2: Page 1128: 1 - 30 all